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The Study of KKM Theorems With Applications to Vector Equilibrium Problems and Implict Vector Variational Inequalities Problems*

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Abstract. In this paper, we establish some equivalence relations between coincidence theorems and KKM-type theorems. We also obtain some new coincidence theorems and fixed point theorems. Applying the KKM-type theorems we obtain the existence theorems of generalized vector equilibrium problems. From these results, some existence theorems of generalized vector implicit variational inequality problems are established in this paper.

Key words: $C(x)$-quasiconvex and $C(x)$-quasiconvexlike, Equilibrium problem, Implict vector variational inequality, KKM property (mapping), Properly quasimonotone, Transfer open (closed), upper (lower) semicontinuous

1. Introduction

In 1929, Knaster et al. [21] established the well-known KKM theorem. Since then, there were many generalizations and applications of KKM theorem; see for example [9, 11, 15–,17, 23, 24, 28–30, 32–39, 42, 44]. Border [8] showed the equivalence relations between Brouwer fixed point theorem, KKM theory and geometric form of minimax theorem, Tarafdar [42] established the equivalence relation between fixed point theorem and KKM theorem, Park [34] studied the equivalence theorems between some KKM theorem, matching theorem, coincidence theorem, and minimax inequality. Recently Lin et al. [28] Lin and Wan [29], Chang and Yen [15] established some generalized KKM theorems and coincidence theorems. In the first part of this paper, we want to establish the equivalence relations between the generalized KKM theorems and coincidence theorems. We establish some new coincidence theorem and fixed point theorem. Our result on fixed point theorem include the results of Tarafdar [17, 41, 42] as a special cases. The coincidence theorem we establish also include recent result of Djafari-Rouhani et al. [17] as a special case.

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In the second part of this paper, we use the generalized KKM theorem in this paper to establish the existence theorems of equilibrium problems.

Let X be a nonempty subset of a topological vector space (in short t.v.s.) E, and $F : X \times X \to R$ be a real valued bifunction such that $F(x, x) \geqslant 0$ for all $x \in X$. Then the scalar equilibrium Problem (in short, EP) is to find $\bar{y} \in X$ such that

$$F(x, \bar{y}) \geqslant 0 \quad \text{for all } x \in X.$$

The EP has many applications in mathematical physics, economics, game theory, and operation research, etc. It contains several problems like optimization, variational inequality, complementarity, Nash equilibrium, and fixed point problems; for detail, see for example [7].

(A) If Z is a t.v.s. with order cone C; that is a closed convex pointed cone, and $F : X \times X \to Z$, then the equilibrium problem (EP) can be generalized in the following ways:

find $\bar{y} \in X$ such that

(1) $F(x, \bar{y}) \in C$ for all $x \in X$; or
(2) $F(x, \bar{y}) \notin (-\text{Int } C)$ for all $x \in X$.

In these cases, (EP) are called vector equilibrium problem (in short, VEP). These problems contains vector optimization, vector variational inequality problem and vector Nash problem as special cases.

(B) Let $F : X \times X - \circ Z$ and $C : X - \circ Z$ be multivalued maps such that $C(x)$ is a closed convex pointed cone for each $x \in X$, (VEP) can be generalized in the following forms.

VEP (1): find $\bar{y} \in X$ such that $F(x, \bar{y}) \subseteq C(\bar{y})$ for all $x \in X$.
VEP (2): find $\bar{y} \in X$ such that $F(x, \bar{y}) \cap C(\bar{y}) \neq \emptyset$ for all $x \in X$.
VEP (3): find $\bar{y} \in X$ such that $F(x, \bar{y}) \not\subseteq (-\text{Int}C(\bar{y}))$ for all $x \in X$.
VEP (4): find $\bar{y} \in X$ such that $F(x, \bar{y}) \cap (-\text{Int}C\ (\bar{y})) = \emptyset$ for all $x \in X$.

Recently, the equilibrium problems in both scalar and vector cases have been extensively studied in many literatures, see [2–6, 12–14, 18–20, 22, 25, 27–29, 31, 40].

In this paper, we consider the above four types of equilibrium problems when F is defined on the product of two different spaces and $C(y)$ is not necessary a closed convex cone for each $y \in Y$. Bianchi and Pini [6] first considered these types of equilibrium problem when $C(y)$ is a constant set for all $y \in Y$. Recently Lin and Wan [29] studied the above four types of equilibrium problems when $C(y)$ is not necessary a cone and is not a constant set. In this paper , we continue the study of Lin and Wan [29]. We study the above four types of equilibrium problems by applying the KKM theorem in Section 3.

As applications of our results, we study the following generalized vector equilibrium problems:

GVEP (1): Find $\bar{y} \in X$ such that

$$g(x, \bar{y}, u) \subseteq C(\bar{y}) \quad \text{for all } u \in \phi(\bar{y}) \text{ and } x \in X;$$

GVEP (2): Find $\bar{y} \in X$ such that for each $x \in X$

$$\text{there exists } u \in \phi(\bar{y}) \quad \text{with } g(x, \bar{y}, u) \cap C(\bar{y}) \neq \emptyset;$$

GVEP (3): Find $\bar{y} \in X$ such that for each $x \in X$

$$\text{there exists } u \in \phi(\bar{y}) \quad \text{with } g(x, \bar{y}, u) \nsubseteq (-\text{Int}C(\bar{y}));$$

GVEP (4): Find $\bar{y} \in X$ such that

$$g(x, \bar{y}, u) \cap (-\text{Int}C(\bar{y})) = \emptyset \quad \text{for all } u \in \phi(\bar{y}) \text{ and all } x \in X,$$

where $g : X \times X \times D - \circ Z, \phi : X - \circ D$ and D is a nonempty subset of topological space Y . Recently Fu and Wan [19] studied the existence theorems of (GVEP (3)).

If g is a single valued function, the above four equilibrium problems are reduced to the following four types of implicit vector variational inequalities problems.

GVEP $(1)'$: Find $\bar{y} \in X$ such that

$$g(x, \bar{y}, u) \in C(\bar{y}) \quad \text{for all } u \in \phi(\bar{y}) \text{ and all } x \in X;$$

GVEP $(2)'$: Find $\bar{y} \in X$ such that for each $x \in X$

$$\text{there exists } u \in \phi(\bar{y}) \quad \text{with } g(x, \bar{y}, u) \in C(\bar{y});$$

GVEP $(3)'$: Find $\bar{y} \in X$ such that for each $x \in X$

$$\text{there exists } u \in \phi(\bar{y}) \quad \text{with } g(x, \bar{y}, u) \notin -\text{Int}C(\bar{y});$$

GVEP $(4)'$: Find $\bar{y} \in X$ such that

$$g(x, \bar{y}, u) \notin -\text{Int}C(\bar{y}) \quad \text{for all } u \in \phi(\bar{y}) \text{ and all } x \in X.$$

Let E be a t.v.s., $L(E, Z) = \{T | T : E \to Z\}$ is a continuous linear operator}, let $u \in L(E, Z), y \in E$, we denote $\langle u, y \rangle$ the evaluation of u at y. If $g(x, y, u) = \langle u, y \rangle + h(x, y)$, then the above equilibrium problem contains the mixed variational inequality problem recently studied by Khanh and Luu [22]. In this paper, we apply the existence theorems of VEP to study the existence theorems of GVEP for both the cases that g is a multivalued map and g is a single valued function.

2. Preliminaries

Let X and Y be nonempty sets. A multivalued map $T : X \multimap Y$ is a function from X into power set of Y. Let $x \in X, B \subseteq Y$ and $y \in Y$, we define $x \in T^{-}(y)$ if and only if $y \in T(x); T^{-}(B) = \{x \in X : \ T(x) \cap B \neq \emptyset\};$ $T^{+}(B) = \{x \in X : T(x) \subseteq B\}$ and $T^{-1}(B)$ will denote either $T^{-}(B)$ or $T^{+}(B)$. For topological space $E, A \subseteq E, A$ is said to be compactly open (compactly closed) if for every nonempty compact subset K of $E, A \cap K$ is open (closed) in K.

Let X and Y be two topological spaces, $T : X - \circ Y, T$ is said to be transfer open [43], if for every $x \in X, y \in T(x)$, there exists an $x' \in X$ such that $y \in \text{int } T(x');$ transfer closed [43], if for every $x \in X, y \notin T(x)$, there exists an $x' \in X$ such that $y \notin \text{cl}T(x'));$ compact if $\overline{T(X)}$ is compact; upper semicontinuous (in short u.s.c.) (resp. lower semicontinuous (in short l.s.c.)) at $x \in X$, if for every open set U in Y with $T(x) \subseteq U$ (resp. $T(x) \cap U \neq \emptyset$), there exists an open neighborhood $V(x)$ of x such that $T(x') \subseteq U$ (resp. $T(x') \cap U \neq \emptyset$) for all $x' \in V(x);$ T is said to be u.s.c. (resp. l.s.c.) on X if T is u.s.c. at every point of X.

LEMMA 2.1 [30]. *Let X and Y be topological spaces and $G : X - \circ Y$ be multivalued map. Then*

(1) *G is transfer closed if and only if* $\bigcap_{x \in X} G(x) = \bigcap_{x \in X} \text{cl}G(x)$;
(2) *G is transfer open if and only if $F : X - \circ Y$ defined by $F(x) = Y \backslash G(x)$ for all $x \in X$ is transfer closed;*
(3) *G is transfer open if and only if* $\bigcup_{x \in X} G(x) = \bigcup_{x \in X} \text{int}G(x)$.

LEMMA 2.2 [26]. *Let X and Y be topological spaces, $T : X - \circ Y$ be a multivaled map. Then the following statements are equivalent:*

(1) *$T^{-} : Y - \circ X$ is transfer open and $T(x) \neq \emptyset$ for all $x \in X$;*
(2) $X = \cup_{y \in Y} \text{int} T^{-}(y)$.

LEMMA 2.3 [40]. *Let X and Y be topological spaces, $T : X - \circ Y$ be a multivalued map. Then T is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$ and any net $\{x_\alpha\}$ in X converges to x, there exists a net $\{y_\alpha\}$ such that* $y_\alpha \in T(x_\alpha)$ with $y_\alpha \to y$.

A convex space [23] X is a nonempty convex set in a vector space with any topology that induces the Euclidean topology on the convex hull of its finite subsets.

Let X be a convex space and Y be a topological space. If $S, T : X - \circ Y$ are multivalued maps such that for each $N \in \langle X \rangle, T(\text{coN}) \subset S(N)$, then S is said to be a generalized KKM mapping w.r.t. T; the multivalued map $T : X - \circ Y$ is said to have the KKM property [15] if $S : X - \circ Y$ is a generalized KKM mapping w.r.t. T such that the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property. We denote by KKM(X, Y) [15] the family of all multivalued maps from X into Y having the KKM property. We denote by $K(X, Y)$ the family of all u.s.c. multivalued maps with compact convex values and $V(X, Y)$ the family of all u.s.c. multivalued maps with compact acyclic values. Any convex set in a Hausdorff t.v.s. is acyclic. Then $K(X, Y) \subseteq V(X, Y)$. In [15], Chang and Yen showed that $V(X, Y) \subseteq$ KKM (X, Y).

LEMMA 2.4 [26]. *Let X be a convex space and Y be a topological space, $T, S : X - \circ Y$ be multivalued maps and $F, H : Y - \circ X$ be defined by $F(y) = X \backslash T^{-}(y)$ and $H(y) = X \backslash S^{-}(y)$. Then the following two statements are equivalent*:

(1) *for each* $y \in Y$, $A \in \langle F(y) \rangle$ *implies* $\text{co}A \subset H(y)$;
(2) *for each* $A \in \langle X \rangle, S(\text{co}A) \subseteq T(A)$.

DEFINITION 2.1 [20]. Let X be a convex subset of a t.v.s and Z be a Hausdorff t.v.s.. Let $C : X - \circ Z$ and $F : X \times X - \circ Z$ be multivalued maps. Given any finite subset $N = \{x_1, x_2, \ldots, x_n\}$ in X and any $x \in \text{co}N$,

(1) F is said to be strong type I C-diagonally quasiconvex in the first argument if for some x_i in N,

$$F(x_i, x) \subseteq C(x);$$

(2) F is said to be strong type II C-diagonally quasiconvex in the first argument if for some x_i in N,

$$F(x_i, x) \cap C(x) \neq \emptyset;$$

(3) F is said to be weak type I C-diagonally quasiconvex in the first argument if for some x_i in N,

$$F(x_i, x) \cap (-\text{Int}C(x)) = \emptyset;$$

(4) F is said to be weak type II C-diagonally quasiconvex in the first argument if for some x_i in N,

$$F(x_i, x) \nsubseteq (-\text{Int}C(x)).$$

THEOREM 2.1 [1]. *Let X and Y be Hausdorff topological spaces, $T : X - \circ Y$ be a multivalued map.*

(1) *If T is an u.s.c. multivalued map with closed values, then T is closed*;

(2) *If X is compact and T is an u.s.c. multivalued map with compact values, then $T(X)$ is compact.*

THEOREM 2.2 [25]. *Let E_1, E_2 and Z be Hausdorff t.v.s., X and Y be nonempty subsets of E_1 and E_2, respectively, $F : X \times Y \times X -\circ Z$ and $S : X -\circ X$.*

(a) *If both S and F are l.s.c., then $T : X \times Y -\circ Z$ which is defined by*

$$T(x, y) = \cup_{u \in S(x)} F(x, y, u) = F(x, y, S(x))$$

is l.s.c on $X \times Y$; and

(b) *If both S and F are u.s.c. multivalued maps with compact values, then T is an u.s.c. multivalued map with compact values.*

3. Some Equivalent KKM-type Theorems and Coincidence Theorems

In this section, we assume that X is a convex space, Y is a Hausdorff topologically space, $T : X -\circ Y$. $S, G : X -\circ Y$ are multivalued maps. We start from the following two new results of [28, 29].

THEOREM 3.1 (Theorem 2.6 [28]). *Suppose that $T \in \mathrm{KKM}(X, Y)$ and that*

(1) *for each $y \in Y, A \in \langle P(y) \rangle$ implies $\mathrm{co}A \subseteq Q(y)$;*
(2) *$P^- : X -\circ Y$ is transfer open and for all $y \in Y, P(y)$ is nonempty;*
(3) *for each compact subset A of $X, \overline{T(A)}$ is compact; and*
(4) *there exists a nonempty compact subset K of Y such that for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that*

$$T(L_N) \backslash K \subseteq \cup \{\mathrm{int} P^-(x) : x \in L_N\}.$$

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{y} \in T(\bar{x})$ and $\bar{x} \in Q(\bar{y})$.

THEOREM 3.2 (Theorem 3.6 [29]). *Suppose that $T \in \mathrm{KKM}(X, Y)$ and that*

(1) *for each $x \in X, T(x) \subseteq S(x)$;*
(2) *for each $A \in \langle X \rangle, S(\mathrm{co}A) \subseteq (A)$;*
(3) *$G : X -\circ Y$ is transfer closed;*
(4) *for each compact subset A of $X, T(A)$ is compact; and*
(5) *there exists a nonempty compact subset K of Y such that for each $N \in < X >$, there exists a compact convex subset L_N of X containing N such that*

$$T(L_N) \cap \bigcap \{\mathrm{cl}G(x) : x \in L_N\} \subseteq K.$$

Then $\bigcap_{x\in X} G(x) \neq \emptyset$.

Proof. Suppose that $\cap_{x\in X}G(x) = \emptyset$. Then for any $y \in Y$, there exists $x \in X$ such that $y \notin G(x)$. Now we define $P', Q' : Y - \circ X$ by $P'(y) = X\backslash G^{-}(y)$ and $Q'(y) = X\backslash S^{-}(y)$ for all $y \in Y$. Then for $y \in Y$, there exists $x \in X$ such that $x \in P'(y)$. By (3), $(P')^{-} : X - \circ Y$ is transfer open. By (2) and Lemma 2.4, for each $y \in Y, A \in \langle P'(y)\rangle$ implies $\mathrm{co}A \subseteq Q'(y)$. By (5), $T(L_N)\backslash K \subseteq \bigcup\{\mathrm{int}(P')^{-}(x) : x \in L_N\}$. Then by Theorem 3.1 that there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{y} \in T(\bar{x})$ and $\bar{x} \in Q'(\bar{y})$. Then $\bar{y} \notin S(\bar{x})$. This contradicts (1). Therefore $\cap_{x\in X}G(x) \neq \emptyset$. □

REMARK 3.1. Theorems 3.1 and 3.2 are equivalent.

Proof. We want to show that Theorem 3.2 implies Theorem 3.1. Under the assumptions of Theorem 3.1. Suppose that for all $x \in X, T(x) \cap Q^{-}(x) = \emptyset$. This implies $T(x) \subset Y\backslash Q^{-}(x)$. Let $H, S : X - \circ Y$ be defined by $H(x) = Y\backslash P^{-}(x)$, and $S(x) = Y\backslash Q^{-}(x)$ for $x \in X$. Then

(1) for each $x \in X, T(x) \subset S(x)$.
(2) $H : X - \circ Y$ is transfer closed.

Since for each $y \in Y, A \in \langle P(y)\rangle$ implies $\mathrm{co}A \subseteq Q(y)$, it follows from Lemma 2.4 that for each $A \in \langle X\rangle$ implies $S(\mathrm{co}A) \subseteq H(A)$. By (4), $T(L_N) \cap \bigcap T\{\mathrm{cl}H(x) : x \in L_N\} \subseteq K$. Hence by Theorem 3.2, we get $\bigcap_{x\in X} H(x) \neq \emptyset$. That is for all $x \in X$, there exists $y \in H(x) = Y\backslash P^{-}(x)$. Then for all $x \in X, y \notin P^{-}(x)$ i.e. $x \notin P(y)$. Thus $P(y) = \emptyset$. This leads to a contradiction. Therefore, there exists $\bar{x} \in X, T(\bar{x}) \bigcap Q^{-}(\bar{x}) \neq \emptyset$. This shows that $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{y} \in T(\bar{x})$ and $\bar{x} \in Q(\bar{y})$. In Theorem 3.2, we see that Theorem 3.1 implies Theorem 3.2. □

The following Theorem is a special case of Theorem 3.2, but it is equivalent to Theorem 3.2.

THEOREM 3.3. *Suppose that* $T \in \mathrm{KKM}(X, Y)$ *and that*

(1) *for each* $x \in X, T(x) \subseteq S(x)$;
(2) *for each* $x \in X, Gx$ *is closed in* Y;
(3) *for any* $N \in < X >, S(\mathrm{co}N) \subset G(N)$;
(4) *for each compact subset* A of X, $\overline{T(A)}$ *is compact; and*
(5) *there exists a nonempty compact subset* K *of* Y *such that for each* $N \in \langle X\rangle$, *there exists a compact convex subset* L_N *of* X *containing* N *such that*

$$T(L_N) \cap \bigcap\{G(x) : x \in L_N\} \subset K.$$

Then $\bigcap\{Gx: \ x \in X\} \neq \emptyset$.

Proof. Since G has closed values, then $G: X - \circ Y$ is transfer closed. Hence by Theorem 3.2, we get $\bigcap\{Gx : x \in X\} \neq \emptyset$. □

THEOREM 3.4. *Theorems 3.2 and 3.3 are equivalent.*

Proof. Under the assumptions of Theorem 3.2. Let $M : X - \circ Y$ be defined by

$$M(x) = \{y \in Y : y \in \text{cl}G(x)\} \quad \text{for } x \in X.$$

Then for each $x \in X, M(x) = \text{cl}G(x)$ is closed. By (4) and Lemma 2.1, $T(L_N) \cap \bigcap T\{\text{cl}G(x) : xL_N\} = T(L_N) \cap \bigcap\{M(x) : x \in L_N\} \subset K$. By Theorem 3.3, $\bigcap_{x\in X} M(x) \neq \emptyset$. That is $\bigcap_{x\in X} \text{cl}G(x) \neq \emptyset$. Since $G : X - \circ Y$ is transfer closed, by Lemma 2.1, $\bigcap_{x\in X} G(x) = \bigcap_{x\in X} \text{cl}G(x) \neq \emptyset$. □

The following theorem is a special case of Theorem 3.1, but it is equivalent to Theorem 3.1.

THEOREM 3.5. *Suppose that* $T \in \text{KKM}(X, Y)$ *and that*

(1) *for each* $y \in Y, A \in \langle P(y)\rangle$ *implies* $\text{co}A \subseteq Q(y)$;
(2) $P^- : X - \circ Y, P^-(x)$ *is open for all* $x \in X$ *and for all* $y \in Y, P(y)$ *is nonempty;*
(3) *for each compact subset* A *of* $X, \overline{T(A)}$ *is compact ; and*
(4) *there exists a nonempty compact subset* K *of* Y *such that for each* $N \in< X >$, *there exists a compact convex subset* L_N *of* X *containing* N *such that*

$$T(L_N)\backslash K \subseteq \cup\{P^-(x) : x \in L_N\}.$$

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ *such that* $\bar{y} \in T(\bar{x})$ *and* $\bar{x} \in Q(\bar{y})$.

THEOREM 3.6. *Theorems 3.1 and 3.5 are equivalent.*

Proof. It is clear that Theorem 3.1 implies Theorem 3.5. Under the assumptions of Theorem 3.1. By (2), $Y = \cup_{x\in X}\text{int}P^-(x)$, therefore, for each $y \in Y$, there exists $x \in X$ such that $y \in \text{int}P^-(x)$. Let $H : Y - \circ X$ be defined by

$$H(y) = \{x \in X : y \in \text{int}P^-(x)\} \quad \text{for } y \in Y.$$

Then $H^-(x) = \text{int}P^-(x)$ is open and for each $y \in Y, H(y) \neq \emptyset$. It is easy to see that $H(y) \subseteq P(y)$ for all $y \in Y$. By (1), for each $y \in Y, A \in \langle H(y)\rangle$ implies $\text{co}A \subset Q(y)$. By (4), for each $y \in T(L_N)\backslash K$, there exists $x \in L_N$

such that $y \in \text{int}P^-(x)$. Therefore $x \in H(y)$ and $y \in H^-(x)$. Hence $T(L_N)\backslash K \subset \bigcup\{H^-(x): x \in L_N\}$. By Theorem 3.5, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{y} \in T(\bar{x})$ and $\bar{x} \in Q(\bar{y})$. □

By Theorem 3.1, we obtain the following coincidence theorem which contains many fixed point theorems and coincidence theorems as special cases.

THEOREM 3.7. *Suppose the conditions* (2) *and* (4) *in Theorem 3.2 are replaced by* (2)′ *and* (4)′ *respectively, where*

(2)′ *for each* $x \in X, P^-(x)$ *contains an open set* $O_x \subseteq Y$ *and* $\cup_{x\in X} O_x = Y$;
(4)′ *there exists a nonempty compact subset* K *of* Y *such that for each* $N \in \langle X \rangle$, *there exists a compact convex subset* L_N *of* X *containing* N *such that*

$$T(L_N)\backslash K \subseteq \cup\{O_x : x \in L_N\}.$$

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ *such that* $\bar{y} \in T(\bar{x})$ *and* $\bar{x} \in Q(\bar{y})$.

Proof. Since $O_x \subseteq P^-(x), O_x$ is open, and $\cup_{x\in X} O_x = Y$. Then $\cup_{x\in X}\text{int}P^-(x) = \bigcup_{x\in X} O_x = Y$. By Lemma 2.2, $P^- : X - \circ Y$ is transfer open and for all $y \in Y, P(y)$ is nonempty. By (4)′, $T(L_N)\backslash K \subseteq \cup\{O_x : x \in L_N\} \subseteq \cup\{\text{int}P^-(x) : x \in L_N\}$. By Theorem 3.1, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{y} \in T(\bar{x})$ and $\bar{x} \in Q(\bar{y})$. □

If $T(x) = \{x\}$ and $P(x) = Q(x)$ for all $x \in X$, then Theorem 3.7 is reduced to the following fixed point theorem.

COROLLARY 3.1. *Suppose that*

(1) *for each* $x \in X, A \in < P(x) >$ *implies* $\text{co}A \subseteq P(x)$;
(2) *for each* $x \in X, P^-(x)$ *contains an open set* $O_x \subset X$ *and* $\cup_{\in X} O_x = X$; *and*
(3) *there exists a nonempty compact subset* K *of* Y *such that for each* $N \in \langle X \rangle$, *there exists a compact convex subset* L_N *of* X *containing* N *such that*

$$L_N\backslash K \subseteq \cup\{O_x : x \in L_N\}.$$

Then there exists $\bar{x} \in X$ *such that* $\bar{x} \in P(\bar{x})$.

THEOREM 3.8. *Theorems 3.1 and 3.7 are equivalent*.

Proof. In Theorem 3.7, we see that Theorem 3.1 implies Theorem 3.7. Under the assumptions of Theorem 3.1. By (2), $X = \cup_{x\in X}\text{int}P^-(x)$. Let

$O_x = \text{int}P^-(x)$. Then O_x is open , $O_x \subseteq P^-(x)$, and $X = \cup_{x \in X} O_x$. By (4), $T(L_N) \backslash K \subseteq \cup\{\text{int}P^-(x) : x \in L_N\} = \cup\{O_x : x \in L_N\}$. Then Theorem 3.1 follows from Theorem 3.7. □

REMARK 3.2. Corollary 3.2 contains Theorem 1 [41], Theorem 1 [42] and Theorem 2.1 [21] as special cases.

REMARK 3.3. Theorems 3.1–3.3, 3.5 and 3.7 are equivalent.

4. Existence Results of General Vector Equilibrium Problems

When dealing with equilibrium problems, the definition of properly quasi-monotone bimap is frequently used (see [5, 6]).

DEFINITION 4.1 [6] Let X, Z be t.v.s., Y a topological space. Let $T : X - \circ Y, F : X \times Y - \circ Z$ and $C : Y - \circ Z$ be multivalued maps. F is said to be properly quasimonotone relatively to T on $X \times Y$ if the map $G : X - \circ Y$,

$$G(x) = \{y \in Y : (x, y) \in F^{-1}(C(y))\}.$$

is a KKM mapping w.r.t. T.

The following Proposition gives a sufficient condition for the properly quasi-monotonicity.

PROPOSITION 4.1. *Let X, Z be* t.v.s., *Y be a topological space. Let $T : X - \circ Y, F : X \times Y - \circ Z$ and $C : Y - \circ Z$. Assume that*

(1) *for all $x \in X$ and $y \in T(x)$, $(x, y) \in F^{-1}(C(y))$; and*
(2) *for any $y \in Y$, $B(y) = \{x \in K : (x, y) \notin F^{-1}(C(y))\}$is convex.*

Then F is properly quasimonotone relative to T.

Proof. Let $G : X - \circ Y$ be defined by

$$G(x) = \{y \in Y : (x, y) \in F^{-1}(C(y))\} \quad \text{for } x \in X.$$

Suppose F is not properly quasimonotone relative to T. Then there exists a finite subset $N = \{x_1, x_2, \ldots, x_n\}$ in X such that $T(\text{co}N) \nsubseteq G(N)$. There exist $\bar{x} \in \text{co}N$ and $\bar{y} \in T(\bar{x})$ such that $\bar{y} \notin G(x_i)$ for all $i = 1, 2, \ldots, n$. Then $(x_i, \bar{y}) \notin F^{-1}(C(\bar{y}))$ for all $i = 1, 2, \ldots, n$. Hence $x_i \in B(\bar{y})$ for all $i = 1, 2, \ldots, n$. By (2), $\bar{x} \in B(\bar{y})$. Thus $(\bar{x}, \bar{y}) \notin F^{-1}(C(\bar{y}))$. This contradicts to condition (1). Therefore, F is properly quasimonotone relative to T. □

REMARK 4.1. There are similar results in [5, 6]. In Proposition 1.1 [5], $C(y) = C$ for all $y \in Y$.

In this section, unless otherwise specify, we assume that X is a convex space, Y a Hausdorff topological space and Z is a Hausdorff t.v.s. We assume $T : X - \circ Y, F : X \times Y - \circ Z$ and $C : Y - \circ Z$ are multivalued maps. Applying Proposition 4.1 and the KKM-type theorems in Section 3, we establish the existence theorems of the four types of (VEP).

THEOREM 4.1. *Let* $T \in \mathrm{KKM}(X, Y)$ *and suppose that*

(1) *for any* $x \in X, F(x, \cdot)$ is l.s.c.*and* $C : Y - \circ Z$ *is closed*;
(2) *for all* $x \in X$ *and* $y \in T(x), F(x, y) \subseteq C(y)$;
(3) *for any* $y \in Y, B(y) = \{x \in X : F(x, y) \nsubseteq C(y)\})$ *is convex*;
(4) *for each compact subset* A of $X, \overline{T(A)}$ *is compact; and*
(5) *there exists a nonempty compact subset* K *of* Y *such that for each* $N \in \langle X \rangle$, *there exists a compact convex subset* L_N of X *containing* N *such that for each* $y \in T(L_N) \backslash K$, *there exists* $x \in L_N$ *with* $F(x, y) \nsubseteq C(y)$.

Then there exists $\bar{y} \in Y$ *such that* $F(x, \bar{y}) \subseteq C(\bar{y}))$ *for all* $x \in X$.

Proof. Let $G : X - \circ Y$ be defined by

$$G(x) = \{y \in Y : F(x, y) \subseteq C(y)\} \quad \text{for } x \in X.$$

Take $F^{-1} := F^+$ in Proposition 4.1. By assumption (2) and (3), for any $N \in \langle X \rangle, T(\mathrm{co}N) \subseteq G(N)$. For any $x \in X, \overline{G(x)}$ is closed; Indeed let $\bar{y} \in \overline{G(x)}$, then there exists a net $\{y_\alpha\}$ in $G(x)$ such that y_α converges to $\bar{y}$. Since $y_\alpha \in G(x), F(x, y_\alpha) \subseteq C(y_\alpha)$. Let $z \in F(x, \bar{y})$. Since for any $x \in X, F(x, \cdot)$ is l.s.c., by Lemma 2.3, there exists a net $\{z_\alpha\}$ such that $z_\alpha \in F(x, y_\alpha)$ with $\{z_\alpha\}$ converges to z. Thus $z_\alpha \in C(y_\alpha)$. Since C is closed, $z \in C(\bar{y})$. Hence $F(x, \bar{y}) \subseteq C(\bar{y})$. That is $\bar{y} \in G(x)$. Therefore $G(x)$ is closed. By Theorem 3.3, $\bigcap_{x \in X} G(x) \neq \emptyset$. Therefore, there exists $\bar{y} \in Y$ such that $F(x, \bar{y}) \subseteq C(\bar{y}))$ for all $x \in X$. □

COROLLARY 4.1. *In Theorem 4.1, if* $T \in \mathrm{KKM}(X, Y)$ *is replaced by* T *is an u.s.c. multivalued map with nonempty compact convex values and suppose that conditions,* (1–3) *and* (5) *of Theorem 4.1 hold.*

Then there exists $\bar{y} \in Y$ *such that* $F(x, \bar{y}) \subseteq C(\bar{y}))$ *for all* $x \in X$.

Proof. Since T is u.s.c. with nonempty compact convex values, $T \in K(X, Y) \subseteq \mathrm{KKM}(X, Y)$ [15]. If A is a compact subset of X, then by Theorem 2.6, $T(A)$ and $\overline{T(A)}$ are compact and Corollary 4.1 follows from Theorem 4.1. □

COROLLARY 4.2. *Suppose that* X *is a Hausdorff convex space,* $F : X \times X - \circ Z$ *and* $C : X - \circ Z$ *is multivalued maps satisfying the following conditions:*

(1) *for any* $x \in X, F(x, \cdot)$ *is l.s.c. and* C *is closed*;
(2) *for all* $x \in X, F(x, x) \subseteq C(x)$;
(3) *for any* $y \in X, B(y) = \{x \in X : F(x, y) \not\subseteq C(y)\}$ *is convex; and*
(4) *there exists a nonempty compact subset* K *of* Y such that for each $N \in \langle X \rangle$, *there exists a compact convex subset* L_N of X *containing* N *such that for each* $y \in L_N \backslash K$, *there exists* $x \in L_N$ *with* $F(x, y) \not\subseteq C(y)$.

Then there exists $\bar{y} \in X$ *such that* $F(x, \bar{y}) \subseteq C(\bar{y})$ *for all* $x \in X$.

Proof. Take $T(x) = \{x\}$ in Corollary 4.1. □

THEOREM 4.2. *Let* X *be a Hausdorff convex space.*
Suppose that conditions (1) *and* (4) *of Corollary 4.2 and* (2′), *where*
(2′) $F : X \times X - \circ Z$ *is strong type I C-diagonally quasiconvex in the first argument.*
Then there exists $\bar{y} \in X$ *such that* $F(x, \bar{y}) \subseteq C(\bar{y}))$ *for all* $x \in X$.

Proof. Let $T(x) = S(x) = \{x\}$ for all $x \in X$ and $G : X - \circ X$ be defined by

$$G(x) = \{y \in X : F(x, y) \subseteq C(y)\} \quad \text{for } x \in X.$$

Suppose that there exists $N = \{x_1, x_2, \ldots, x_n\} \in \langle X \rangle$ such that co$N = T($co$N) \not\subseteq G(N)$. Then there exists $\bar{x} \in$ co$N$ such that $\bar{x} \notin G(x_i)$ for all $i = 1, 2, \ldots, n$. That is $F(x_i, \bar{x}) \not\subseteq C(\bar{x})$ for all $i = 1, 2, \ldots, n$. But F is strong type I P-diagonally quasiconvex in the first argument (2′), $F(x_i, \bar{x}) \subseteq C(\bar{x})$ for some $x_i \in N$. This leads to a contradiction. Hence for each $N \in \langle X \rangle$, co$N \subseteq G(N)$. By (1), $G(x)$ is closed for all $x \in X$. Therefore the conclusion of Theorem 4.2 follows from Theorem 3.3. □

DEFINITION 4.2. Let $C : X - \circ Z$ be a multivalued map such that $C(y)$ is convex cone for all $y \in Y$. We say that

(a) For each $y \in Y, x - \circ F(x, y)$ is $C(y)$-quasiconvex [20] if $x_1, x_2 \in X$, and $\lambda \in [0, 1]$, then either

$$F(x_1, y) \subseteq F(\lambda x_1 + (1 - \lambda)x_2, y) + C(y).$$
$$\text{or} \quad F(x_2, y) \subseteq F(\lambda x_1 + (1 - \lambda)x_2, y) + C(y).$$

(b) For each $y \in Y, x - \circ F(x, y)$ is $C(y)$-quasiconvex-like [3] if for $x_1, x_2 \in X$, and $\lambda \in [0, 1]$, then either

$$F(\lambda x_1 + (1 - \lambda)x_2, y) \subseteq F(x_1) - C(y).$$
$$\text{or} \quad F(\lambda x_1 + (1 - \lambda)x_2, y) \subseteq F(x_2) - C(y).$$

REMARK 4.2. If $C(y)$ is a convex cone and $F(\cdot, y)$ is $C(y)$-quasiconvex, then the set $B(y) = \{x \in X : F(x, y) \not\subseteq C(y)\}$ is convex.

Proof. (a) To prove $B(y) = \{x \in X : F(x, y) \not\subseteq C(y)\}$ is convex.

Let $x_1, x_2 \in B(y)$, then $F(x_1, y) \not\subseteq C(y)$ and $F(x_1, y) \not\subseteq C(y)$. We want to show that $\lambda x_1 + (1-\lambda)x_2 \in B(y)$ for all $\lambda \in [0, 1]$. Suppose there exists $\lambda_0 \in [0, 1]$ such that $F(\lambda_0 x_1 + (1-\lambda_0)x_2, y) \subseteq C(y)$. Since $F(\cdot, y)$ is $C(y)$-quasiconvex, either $F(x_1, y) \subseteq C(y)$ or $F(x_2, y) \subseteq C(y)$. This leads to a contradiction. Hence for all $\lambda \in [0, 1], F(\lambda x_1 + (1-\lambda)x_2, y) \not\subseteq C(y)$ and $B(y)$ is convex.

By using Theorem 3.3 and Theorem 4.1, we establish the following theorem.

THEOREM 4.3. *Suppose that* $T \in \mathrm{KKM}(X, Y)$, (1), (4), (5) *of Theorem 4.1 and*

(a) *for all* $x \in X$ *and* $y \in T(x), A(x, y) \subseteq C(y)$;
(b) *for all* $(x, y) \in X \times Y, A(x, y) \subseteq C(y)$ *implies* $F(x, y) \subseteq C(y)$;
(c) *for any* $y \in Y, B(y) = \{x \in X : A(x, y) \not\subseteq C(y)\}$ *is convex*;

Then there exists $\bar{y} \in Y$ *such that* $F(x, \bar{y}) \subseteq C(\bar{y})$, *for all* $x \in X$.

Proof. Let $G, H : X - \circ Y$ be defined by $G(x) = \{y \in Y : A(x, y) \subseteq C(y)\}$ and $H(x) = \{y \in Y : F(x, y) \subseteq C(y)\}$ for $x \in X$. By (1), $H(x)$ is closed for all $x \in X$. By (b), for any $x \in X, G(x) \subseteq H(x)$. By (a) ,(c) and Proposition 4.1 that for all $N \in \langle X \rangle, T(\mathrm{co}N) \subseteq G(N)$. By (c), $T(L_N) \cap \bigcap \{H(x) : x \in L_N\} \subset K$. Then by Theorem 3.3, $\bigcap_{x \in X} H(x) \neq \emptyset$, i.e. there exists $\bar{y} \in Y$ such that $F(x, \bar{y}) \subseteq C(\bar{y})$, for all $x \in X$. □

DEFINITION 4.3 [18]. Let $H : X - \circ Z$ be a multivalued map. H is said to be properly quasiconvex if for every $x, y \in K, t \in [0, 1]$, and $u \in H(x), v \in H(y)$, there exists $z \in H(tx_1 + (1-t)x_2)$ such that either $z \leqslant u$ or $z \leqslant v$.

LEMMA 4.1 [18]. *Let* $H : X - \circ Z$ *be a multivalued mapping. Then* H *is properly quasiconvex if and only if for any* $x_i \in K, z_i \in H(x_i), t_i > 0$, *for* $i = 1, 2, \ldots, n, \sum_{i=1}^{n} t_i = 1$, *there exist* $z \in F(t_1 x_1 + \cdots + t_n x_n)$ *and some* i *such that* $z \leqslant z_i$.

THEOREM 4.4. *In Theorem 4.1, if condition* (3) *is replaced by* (3)$'$, *then we have the same conclusion, where*

(3)$'$ *for any* $y \in Y, x - \circ F(x, y)$ *is properly quasiconvex.*

Proof. Let $G : X - \circ Y$ be defined by

$$G(x) = \{y \in Y : F(x, y) \subseteq C(y)\} \quad \text{for } x \in X.$$

By (2), (3)$'$ and following the same arguments as in Theorem 2 of [18], we can show that for all $N \in \langle X \rangle, T(\text{co}N) \subseteq G(N)$. Then the conclusion of Theorem 4.4 follows from Theorem 3.3. □

REMARK 4.3. Theorem 4.4 is different from Theorem 2 [18]. In Theorem 2 [18], the multivalued map $T : X - \circ D$ is u.s.c. with nonempty compact convex values, X is a nonempty compact convex set, and $G(x)$ is closed for all $x \in X$.

THEOREM 4.5. *Suppose* $T \in \text{KKM}(X, Y)$ *and*

(1) *for any* $x \in X$, $y - \circ F(x, y)$ *is u.s.c. with compact values and* $C : Y - \circ Z$ *is closed*;
(2) *for all* $x \in X$ *and* $y \in T(x), F(x, y) \cap C(y) \neq \emptyset$;
(3) *for any* $y \in Y, B(y) = \{x \in X : F(x, y) \cap C(y) = \emptyset\}$ *is convex*;
(4) *for each compact subset* A of X, $\overline{T(A)}$ *is compact; and*
(5) *there exists a nonempty compact subset* K of Y *such that for each* $N \in \langle X \rangle$, *there exists a compact convex subset* L_N *of* X *containing* N *such that for each* $y \in T(L_N) \backslash K$, *there exists* $x \in L_N$ *with* $F(x, y) \cap C(y) = \emptyset$.

Then there exists $\bar{y} \in Y$ *such that* $F(x, \bar{y}) \cap C(\bar{y}) \neq \emptyset$; *for all* $x \in X$.

Proof. Let $G : X - \circ Y$ be defined by

$$G(x) = \{y \in Y : F(x, y) \cap C(y) \neq \emptyset\} \quad \text{for } x \in X.$$

Take $F^{-1} := F^{-}$ in Proposition 4.1. By (2), (3) and Proposition 4.1, for any $N \in \langle X \rangle, T(\text{co}N) \subseteq G(N)$. By (1) and following the same argument as in [3], $G(x)$ is closed for each $x \in X$. Then by Theorem 3.3, $\cap_{x \in X} G(x) \neq \emptyset$. Therefore, there exists $\bar{y} \in Y$ such that $F(x, \bar{y}) \cap C(\bar{y}) \neq \emptyset$ for all $x \in X$. □

REMARK 4.4. Following the method of Ansari and Yao [3], we can show that if for each $y \in Y, F(\cdot, y)$ is $C(y)$-quasiconvex-like and $C(y)$ is a convex cone for each $y \in Y$. Then the set $B(y) = \{x \in X : F(x, y) \cap C(y) = \emptyset\}$ is convex.

THEOREM 4.6. *Let* X *be a Hausdorff convex space, let* $F : X \times X - \circ Z$ *and* $C : X - \circ Z$ *be multivalued maps. Suppose that*

(1) *of Theorem* 4.5 *and*
(2)$'$ F *is strong type II C-diagonally quasiconvex in the first argument; and*

(3)$'$ *there exists a nonempty compact subset K of X such that for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that for each $y \in L_N \backslash K$, there exists $x \in L_N$* with $F(x, y) \cap C(y) = \emptyset$.

Then there exists $\bar{y} \in X$ such that $F(x, \bar{y}) \cap C(\bar{y}) \neq \emptyset$, *for all $x \in X$.*

Proof. Let $T(x) = \{x\}$ for $x \in X$ and $G : X - \circ X$ be defined by

$$G(x) = \{y \in X : F(x, y) \cap C(y) \neq \emptyset\} \quad \text{for } x \in X.$$

Suppose that there exists $N = \{x_1, x_2, \ldots, x_n\} \in \langle X \rangle$ such that $\text{co}N = T(\text{co}N) \not\subseteq G(N)$. Then there exists $\bar{x} \in \text{co}N$ such that $\bar{x} \notin G(x_i)$ for all $i = 1, 2, \ldots, n$. That is $F(x_i, \bar{x}) \cap C(\bar{x}) = \emptyset$ for all $i = 1, 2, \ldots, n$. But by (2)$'$, $F(x_i, \bar{x}) \cap C(\bar{x}) \neq \emptyset$ for some $x_i \in N$. This leads to a contradiction. Hence for each $N \in \langle X \rangle, \text{co}N \subseteq G(N)$. Therefore the conclusion of Theorem 4.6 follows from Theorem 3.3.

Applying Remark 3.3 of Lin [28], we establish the following theorem

THEOREM 4.7. *Suppose that $T \in KKM(X, Y)$ and*

(1) *for any $x \in X, y - \circ F(x, y)$ is u.s.c. with compact values and $W : Y - \circ Z$ is u.s.c., where $W(y) = Z \backslash (-\text{Int}C(y))$*for all $y \in Y$;
(2) *for all $x \in X$ and $y \in T(x), F(x, y) \not\subseteq (-\text{Int}C(y))$;*
(3) *for any $y \in Y, B(y) = \{x \in X : \underline{F(x, y)} \subseteq (-\text{Int}C(y))\}$ is convex;*
(4) *for each compact subset A of X, $\overline{T(A)}$ is compact; and*
(5) *there exists a nonempty compact subset K of Y such that for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that for each $y \in T(L_N) \backslash K$, there exists $x \in L_N$ with $F(x, y) \subseteq (-\text{Int}C(y))$.*

Then there exists $\bar{y} \in Y$ such that $F(x, \bar{y}) \not\subseteq (-\text{Int}C(\bar{y}))$ for all $x \in X$.

Proof. Let $G : X - \circ Y$ be defined by

$$G(x) = \{y \in Y : F(x, y) \not\subseteq (-\text{Int}C(y)\} \quad \text{for } x \in X.$$

By (2), (3) and Proposition 4.1, for all $N \in \langle X \rangle, T(\text{co}N) \subseteq G(N)$.

By (1) and following the same argument as in Theorem 2.1 of Ansari and Yao [3], for each $x \in X, G(x)$ is closed. By Theorem 3.3 , $\cap_{x \in X} G(x) \neq \emptyset$. Therefore, there exists $\bar{y} \in Y$ such that $F(x, \bar{y}) \not\subseteq (-\text{Int}C(\bar{y}))$, for all $x \in X$. □

REMARK 4.5. Following the method of Ansari and Yao [3], we can show that if F is $C(y)$-quasiconvexlike and $C(y)$ is a convex cone for each $y \in Y$, then the set $B(y) = \{x \in X : F(x, y) \subseteq (-\text{Int}C(y))\}$ is convex.

THEOREM 4.8. *Let X be a Hausdorff convex space, let $F : X \times X - \circ Z$ and $C : X - \circ Z$ be multivalued maps. Suppose that*

(1) *of Theorem 4.7 and*
(2)$'$ *F is weak type II C-diagonally quasiconvex in the first argument for each fixed y; and*
(3)$'$ *there exists a nonempty compact subset K of X such that for each $N \in \langle X \rangle$, there exists a compact convex subset L_N* of *X containing N such that for each $y \in L_N \backslash K$, there exists $x \in L_N$ with $F(x, y) \subseteq (-\mathrm{Int}C(y))$.*

Then exists $\bar{y} \in X$ such that $F(x, \bar{y}) \nsubseteq (-\mathrm{Int}C(\bar{y}))$ for all $x \in X$.

Proof. Let $T(x) = \{x\}$ for $x \in X$ and $G : X - \circ X$ be defined by

$$G(x) = \{y \in X : F(x, y) \nsubseteq (-\mathrm{Int}C(y))\} \quad \text{for } x \in X.$$

By (1) and following the same argument as in Theorem 4.1, $G(x)$ is closed. By (2)$'$ and follows the same argument as in Theorem 4.2, we show that for each $N \in \langle X \rangle, \mathrm{co}N = T(\mathrm{co}N) \subseteq G(N)$. Therefore the conclusion of Theorem 4.8 follows from Theorem 3.3.

THEOREM 4.9. *Suppose that $T \in \mathrm{KKM}(X, Y)$,* (4) *of Theorem 4.7 and*

(1)$'$ *for any $x \in X, y - \circ F(x, y)$ is l.s.c. and $W : Y - \circ Z$ is u.s.c, where $W(y) = Z \backslash (-\mathrm{Int}C(y))$ for all $y \in Y$;*
(2)$'$ *for all $x \in X$ and $y \in T(x), F(x, y) \cap (-\mathrm{Int}C(y)) = \emptyset$;*
(3)$'$ *for any $y \in Y, B(y) = \{x \in X : F(x, y) \cap (-C(y)) \neq \emptyset\}$is convex; and*
(5)$'$ *there exists a nonempty compact subset K of Y such that for each $N \in \langle X \rangle$,there exists a compact convex subset L_N of X containing N such that for each $y \in T(L_N) \backslash K$, there exists $x \in L_N$ with $F(x, y) \cap (-\mathrm{Int}C(y)) \neq \emptyset$.*

Then there exists $\bar{y} \in Y$ such that $F(x, y) \cap (-\mathrm{Int}C(y)) = \emptyset$ for all *$x \in X$.*

Proof. Let $G : X - \circ Y$ be defined by

$$G(x) = \{y \in X : F(x, y) \cap (-\mathrm{Int}C(y)) = \emptyset\} \quad \text{for } x \in X.$$

By (2)$'$ (3)$'$ and Proposition 4.1, for all $N \in \langle X \rangle, T(\mathrm{co}N) \subseteq G(N)$.

By (1)$'$ $G(x)$ is closed for all $x \in X$. By Theorem 3.3, there exists $\bar{y} \in Y$ such that $F(x, \bar{y}) \cap (-\mathrm{Int}C(\bar{y})) = \emptyset$ for all $x \in X$.

THEOREM 4.10. *Let X be a Hausdorff convex space, let $F : X \times X - \circ Z$ and $C : X - \circ Z$ be multivalued maps. Suppose that*

(1) *for any $x \in X, y - \circ F(x, y)$ is l.s.c. and $W : Y - \circ Z$ is u.s.c., where $W(y) = Z \backslash (-\mathrm{Int}C(y))$* for all *$y \in Y$;*
(2) *F is weak type I C-diagonally quasiconvex in the first argument; and*

(3) *there exists a nonempty compact subset K of X such that for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that for each $y \in L_N \backslash K$, there exists $x \in L_N$ with $F(x,y) \cap (-\mathrm{Int}C(y)) \neq \emptyset$.*

Then there exists $\bar{y} \in X$ such that $F(x,\bar{y}) \cap (-\mathrm{Int}C(\bar{y})) = \emptyset$ for all $x \in X$.

Proof. Let $T(x) = \{x\}$ for $x \in X$ and $G : X - \circ X$ be defined by

$$G(x) = \{y \in X : F(x,y) \cap (-\mathrm{Int}C(y)) = \emptyset\} \quad \text{for } x \in X.$$

By (1), $G(x)$ is closed for all $x \in X$. With the similar argument as Theorem 4.2, we show that for each $N \in \langle X \rangle, \mathrm{co}N = T(\mathrm{co}N) \subseteq G(N)$. Therefore Theorem 4.10 follows from Theorem 3.3. □

The rest of this section, let X be a Hausdorff convex space, Y a Hausdorff topological space, D be a nonempty subset of Y, Z be a Hausdorff t.v.s. As simple consequences of (VEP), we establish the existence theorems of (GVEP).

THEOREM 4.11. *Suppose that $g : X \times X \times D - \circ Z, \phi : X - \circ D$ and*

(1) *for each fixed $x \in X, (y,u) - \circ g(x,y,u)$ and ϕ are l.s.c. and C is closed;*
(2) *for each $y \in X$ and $u \in \phi(y), g(y,y,u) \subseteq C(y)$;*
(3) *for each fixed $y \in X, x - \circ g(x,y,\phi(y))$ is $C(y)$-quasiconvex; and*
(4) *there exists a nonempty compact subset K of X such that for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that for each $y \in L_N \backslash K$, there exists $x \in L_N$* and *$u \in \phi(y)$ such that $g(x,y,u) \nsubseteq C(y)$.*

Then there exists $\bar{y} \in X$ such that $g(x,\bar{y},u) \subseteq C(\bar{y}))$ for all $u \in \phi(\bar{y})$ and all $x \in X$.

Proof. Let $F(x,y) = g(x,y,\phi(y)) = \cup_{u \in \phi(y)} g(x,y,u)$. By (1) and Theorem 2.2, $F(x,y)$ is l.s.c. By (3), the multivalued map $x - \circ F(x,y)$ is $C(y)$-quasiconvex. Then by Remark 4.2, the set $\{x \in X : F(x,y) \nsubseteq C(y)\}$ is convex. By Corollary 4.2, then there exists $\bar{y} \in X$ such that $g(x,\bar{y},u) \subseteq C(\bar{y})$ for all $x \in X$ and $u \in \phi(\bar{y})$. □

In Theorem 4.11, if $g : X \times X \times D - \circ Z$ is a single value function, we have the following existence theorem of generalized vector implicit vector variational inequality.

COROLLARY 4.3. *Suppose that D, g and ϕ be the same as Theorem 4.11 and*

(1) *ϕ is l.s.c. and for each fixed $x \in X, (y, u) \to g(x, y, u)$ is a continuous function and C is closed and $C(y)$ is a convex cone for each $y \in Y$;*
(2) *for each $y \in X$ and $u \in \phi(y), g(y, y, u) \in C(y)$;*
(3) *for each fixed $y \in X, x - \circ g(x, y, \phi(y))$is $C(y)$-quasiconvex; and*
(4) *there exists a nonempty compact subset K of X such that for each $N \in< X >$, there exists a compact convex subset L_N of X containing N such that for each $y \in L_N \backslash K$, there exists $x \in L_N, u \in \phi(y)$ such that $g(x, y, u) \in C(y)$.*

Then there exists $\bar{y} \in X$ such that $g(x, \bar{y}, u) \in C(\bar{y}))$ for all $u \in \phi(\bar{y})$ and all $x \in X$.

THEOREM 4.12. *Suppose that $g : X \times X \times D - \circ Z, \phi : X - \circ D$ and*

(1) *for each fixed $x \in X, (y, u) - \circ g(x, y, u)$and ϕ are u.s.c. with compact values and $C : Y - \circ Z$ is closed*;
(2) *for all $y \in X$ and $u \in \phi(y), g(y, y, u) \cap C(y) \neq \emptyset$;*
(3) *for each fixed $y \in X, x - \circ g(x, y, \phi(y))$is $C(y)$-quasiconvex-like and $C(y)$ is a closed convex cone for each $y \in Y$; and*
(4) *there exists a nonempty compact subset K of X such that for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that for each $y \in L_N \backslash K$, there exists $x \in L_N$ with $g(x, y, \phi(y)) \cap C(y) = \emptyset$.*

Then there exists $\bar{y} \in X$ such that for each $x \in X$, there exists *$u \in \phi(\bar{y})$with $g(x, \bar{y}, u) \cap C(\bar{y}) \neq \emptyset$.*

Proof. Let $F(x, y) = g(x, y, \phi(y)) = \cup_{u \in \phi(y)} g(x, y, u)$. By (1) and Theorem 2.7, for each $x \in X, F(x, y)$ is u.s.c. with compact values. Since for each $y \in X, x - \circ F(x, y)$ is $C(y)$-quasiconvex-like and $C(y)$ is a closed convex cone, by (3) and Remark 4.4, the set $\{x \in X : F(x, y) \cap C(y) \neq \phi\}$ is convex. By Theorem 4.5 with $T(x) = \{x\}$ for all $x \in X$, there exists $\bar{y} \in X$ such that $F(x, \bar{y}) = g(x, \bar{y}, \phi(\bar{y})) \cap C(\bar{y}) \neq \emptyset$ for all $x \in X$. Therefore for each $x \in X$, there exists $u \in \phi(\bar{y})$ such that $g(x, \bar{y}, u) \cap C(\bar{y}) \neq \emptyset$. □

For the special cases of Theorem 4.12, we have the following existence theorem of generalized implicit vector variational inequality.

COROLLARY 4.4. *Let $g : X \times X \times D \to Z$ be a function satisfying the following conditions*:

(1) *for each fixed $x \in X, (y, u) \to g(x, y, u)$ is continuous, ϕ is u.s.c. with compact values and C is closed*;
(2) *for any $y \in X$ and $u \in \phi(y), g(y, y, u) \in C(y)$*;

(3) *for each fixed* $y \in X, x - \circ g(x, y, \phi(y))$ *is* $C(y)$*-quasiconvex-like and* $C(y)$ *is a closed convex cone for each* $y \in Y$*; and*
(4) *there exists a nonempty compact subset* K *of* X *such that for each* $N \in <X>$ *there exists a compact convex subset* L_N *of* X *containing* N *such that for each* $y \in L_N \backslash K$, *there exists* $x \in L_N$ *such that* $g(x, y, u) \notin C(y)$ *for all* $u \in \phi(y)$.

Then there exists $\bar{y} \in X$ *such that for each* $x \in X$, *there exists* $u \in \phi(\bar{y})$*such that* $g(x, \bar{y}, u) \in C(\bar{y})$.

THEOREM 4.13. *Suppose that* $g, h : X \times X \times D - \circ Z, \phi : X - \circ D$ *and*

(1) *for each fixed* $x \in X, (y, u) - \circ g(x, y, u)$ *and* ϕ *are u.s.c. with compact values and* $W : X - \circ Z$ *is u.s.c., where* $W(y) = Z \backslash (-\mathrm{Int}C(y))$ *for all* $y \in Y$*;*
(2) *for all* $y \in X$, *there exists* $u \in \phi(y)$ *such that* $g(y, y, u) \nsubseteq -\mathrm{Int}C(y)$*;*
(3) *for each fixed* $y \in X, x - \circ g(x, y, \phi(y))$ *is* $C(y)$*-quasiconvex-like and* $C(y)$ *is a closed convex cone for each* $y \in Y$*; and*
(4) *there exists a nonempty compact subset* K *of* X *such that for each* $N \in \langle X \rangle$, *there exists a compact convex subset* L_N *of* X *containing* N *such that for each* $y \in L_N \backslash K$, *there exists* $x \in L_N$ *with* $g(x, y, \phi(y)) \subseteq (-\mathrm{Int}C(y))$.

Then there exists $\bar{y} \in X$ *such that for each* $x \in X$, *there exists* $u \in \phi(\bar{y})$ *with* $g(x, \bar{y}, u) \nsubseteq (-\mathrm{Int}C(\bar{y}))$.

Proof. Let $F(x, y) = g(x, y, \phi(y)) = \cup_{u \in \phi(y)} g(x, y, u)$. Let $G : X - \circ X$ be defined by

$$G(x) = \{y \in X : F(x, y) = g(x, y, \phi(y)) \nsubseteq (-\mathrm{Int}C(y))\} \quad \text{for } x \in X.$$

By Theorem 2.2 and Theorem 4.7, we can prove that Theorem 4.13. □

For the special case of Theorem 4.13, we have the following existence theorem of generalized vector implicit variational inequality.

COROLLARY 4.5. *Suppose that* $g : X \times X \times D \to Z$ *and* $\phi : X - \circ Z$ *and*

(1) *for each fixed* $x \in X, (y, u) \to g(x, y, u)$ *is continuous,* ϕ *is u.s.c. with compact values and* $W : X - \circ Z$ *is u.s.c., where* $W(y) = Z \backslash (-\mathrm{Int}C(y))$ *for all* $y \in Y$;
(2) *for all* $y \in X$ *there exist* $u \in \phi(y)$*such that* $g(y, y, u) \notin (-\mathrm{Int}C(y))$;
(3) *for each fixed* $y \in X, x - \circ g(x, y, \phi(y))$*is* $C(y)$*-quasiconvex-like and* $C(y)$ *is a closed convex cone for each* $y \in Y$;

(4) *there exists a nonempty compact subset K of X such that for each $N \in \langle X \rangle$ there exists a compact convex subset L_N of X containing N such that for each $y \in L_N \backslash K$, there exists $x \in L_N$ with $g(x, y, u) \in (-\text{Int}C(y))$ for all $u \in C(y)$.*

Then there exists $\bar{y} \in X$ such that for each $x \in X$, there exists $u \in \phi(\bar{y})$ with $g(x, \bar{y}, u) \notin (-\text{Int } C(\bar{y}))$.

THEOREM 4.14. *Suppose that $g : X \times X \times D - \circ Z, \phi : X - \circ D$ and*

(1) *for each fixed $x \in X$, the multivalued maps $(y, u) - \circ g(x, y, u)$ and ϕ are l.s.c. and $W : X - \circ Z$ is u.s.c., where $W(y) = Z \backslash (-\text{Int}C(y))$for all $y \in Y$;*
(2) *$g(x, y, \phi(y))$ is weak type I C-diagonally quasiconvex in the first argument; and*
(3) *there exists a nonempty compact subset K of X such that for each $N \in < X >$, there exists a compact convex subset L_N of X containing N such that for each $y \in L_N \backslash K$, there exists $x \in L_N$ with $g(x, y, \phi(y)) \cap (-\text{Int}C(y)) \neq \emptyset$.*

Then there exists $\bar{y} \in X$ such that $g(x, \bar{y}, u) \cap (-\text{Int}C(\bar{y})) = \emptyset$ for all $u \in \phi(\bar{y})$ and all $x \in X$.

Proof. Let $F(x, y) = g(x, y, \phi(y)) = \cup_{u \in \phi(y)} g(x, y, u)$. Let $G : X - \circ X$ be defined by

$$G(x) = \{y \in X : F(x, y) = g(x, y, \phi(y)) \cap (-\text{Int}C(y)) = \emptyset\} \quad \text{for } x \in X.$$

Theorem 4.14 follows from Theorems 2.7 and 4.10.

COROLLARY 4.6. *Suppose that $g : X \times X \times D \to Z, \phi : X - \circ Z$ and*

(1) *for each fixed $x \in X, (y, u) \to g(x, y, u)$ is continuous, ϕ is l.s.c. and $W : X - \circ Z$ is u.s.c., where $W(y) = Z \backslash\ (-\text{Int}C(y))$ for all $y \in Y$;*
(2) *$g(x, y, \phi(y))$ is weak type I C-diagonally quasiconvex in the first argument; and*
(3) *there exists a nonempty compact subset K of X such that for each $N \in < X >$, there exists a compact convex subset L_N of X containing N such that for each $y \in L_N \backslash K$, there exists $x \in L_N, u \in \phi(y)$ with $g(x, y, u) \in (-\text{Int}C(y))$.*

Then there exists $\bar{y} \in X$ such that $g(x, \bar{y}, u) \notin (-\text{Int}C(\bar{y}))$ for all $u \in \phi(\bar{y})$ and all $x \in X$.

REMARK 4.6.

(1) Corollaries 4.4 and 4.5 are different from Theorems 3 and 4 [19].
(2) If $g(x, y, u) =< u, \eta(y, x) > +h(x, y)$

where $h : X \times X \to Z$ and $\eta : X \times X \to X$, $u \in L(X, Z) = \{T | T : X \to Z$ is a continuous linear operator$\}$. Then Corollaries 4.5 and 4.6 are the existence theorems of mixed generalized vector variational inequality problems recently studied by Khanh and Luu. [22].

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